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A Berry and a Russell without self-reference

The paradoxes of definability - Berry's paradox, Richard's paradox, and König's paradox - all exhibit some form of circularity, or self-reference understood in a suitably broad sense. Consider Berry's paradox, for example. There are denumerably many positive integers, but only finitely many phrases of English with fewer than twenty eight syllables. So there is an integer which is the least positive integer not denoted by an English phrase with fewer than twenty eight syllables. Now consider the English phrase formed by the last sixteen words of the previous sentence. This phrase - call it the Berry phrase - denotes a positive integer. But the Berry phrase has twenty seven syllables. So the least positive integer which cannot be denoted by a phrase with less than twenty eight syllables is denoted by a phrase with twenty seven syllables - contradiction. Observe that the Berry phrase involves quantification over a certain domain of English phrases which contains the Berry phrase itself. Here is the circularity - or self-reference, in the sense that the Berry phrase makes indirect reference to itself.

We may construct new versions of the definability paradoxes where the self-reference is more explicit. For example, consider the paradox generated by the following phrases written on the board in room 101:

A. The ratio of the circumference of a circle to its diameter.

B. The positive square root of 36.

C. The sum of the numbers denoted by expressions on the board in room 101.[1]

Are there definability paradoxes without self-reference? We can certainly say that there are pathological denoting expressions whose pathology does not turn on circularity or self-reference. Consider the following infinite chain of expressions:

A_1. The positive integer denoted by A_2.

A_2. The positive integer denoted by A_3.

.
.
.

A_n The positive integer denoted by A_{n+1}.

.
.
.

But we cannot be said to have a paradox or antinomy here, because no contradiction is forced upon us.

Let us work our way towards a paradox of definability without self-reference. Let E_1, E_2, ... be an arbitrary finite or denumerable list of denoting expressions. Some of these expression may denote positive integers (i.e. 1, 2, 3, ...). The expression 'max(E_1,E_2,...)' denotes the largest positive integer denoted by an expression on the list. If denumerably many distinct positive integers are denoted by expressions on the list (so that there is no largest positive integer denoted), 'max(E_1,E_2,...)' denotes ω, the first infinite ordinal; and if no expression on the list denotes a positive integer, 'max(E_1,E_2,...)' denotes 0. So, for example, max('London', 'the only even prime', 'the successor of 2', 'red') = 3; max('London', 'New York', 'L.A.') = 0; and max('one', 'three', 'five', ... 'thirty one', ...) = ω.

Now consider the following infinite list of denoting expressions:

D_1. 1+max(D_2,...,D_n,...).

D_2. 1+max(D_3,...,D_n,...).
.
.
.
D_n.   1+max(D_{n+1},...,D_{n+i},...).

D_{n+1}.1+max(D_{n+2},...,D_{n+i},...).
.
.
.

Suppose towards a contradiction that for some arbitrary n, D_n denotes a positive integer, say p. Now D_n is given by:

D_n.   1+max(D_{n+1},..., D_{n+i},...).

Since D_n denotes p, max(D_{n+1},..., D_{n+i},...) = p-1. So there is an expression D_k among D_{n+1}, ..., D_{n+i}, ... which denotes p-1.[2] D_k is given by:

D_k.   1+max(D_{k+1},..., D_{k+i},...).

Since D_k denotes p-1, max(D_{k+1},..., D_{k+i},...) = p-2. So there is an expression D_l among D_{k+1}, ..., D_{k+i}, ... which denotes p-2. And so on. Continuing in this way (for p-3 more steps), we obtain an expression D_z which denotes 1. D_z is given by:

D_z. 1+max(D_{z+1},..., D_{z+i},...).

Since D_z denotes 1, max(D_{z+1},..., D_{z+i},...) = 0. By the definition of 'max', none of D_{z+1}, ..., D_{z+i}, ... denote a positive integer. In particular,

(i) D_{z+1} does not denote a positive integer.

Now D_{z+1} is given by:

D_{z+1}. 1+max(D_{z+2},...,D_{z+i},...).

Since none of D_{z+2}, ..., D_{z+i}, ... denote a positive integer, max(D_{z+2},...,D_{z+i},...) = 0. But then D_{z+1} denotes 1+0. That is,

(ii) D_{z+1} denotes a positive integer, namely 1.

From (i) and (ii), we obtain a contradiction.

By our reductio argument, we have shown that no D_n denotes a positive integer, for any n. So for all n, max(D_n,...,D_{n+i},...) = 0. In particular, then, max(D_2,...,D_n,...)=0; max(D_3,...,D_n,...)=0; and so on. But then D_1 denotes 1+0; D_2 denotes 1+0; and in general, D_n denotes 1+0. To sum up: no D_n denotes a positive integer, and every D_n denotes a positive integer (namely 1). We are landed in paradox.

Observe that this paradox does not display any self-reference: each denoting expression makes reference only to phrases further down the list. This suggests that an adequate solution to the paradoxes of definability must do more than avoid circularity or self-reference - the roots of these paradoxes go deeper.

We can draw the same moral about the set-theoretical paradoxes. It is true that all the standard set-theoretical paradoxes exhibit circularity and self-reference, broadly construed. For example, standard versions of Russell's paradox turn on self-membership and circularity. We take the set of all non-self-membered sets, and ask whether or not this set itself is self-membered; or we consider the predicate 'non-self-membered extension', and ask whether or not the extension of this very predicate is self-membered. Or consider the following simple version of the Russell. Suppose the following two predicates are the only expressions written on the board in room 102:

(A) moon of the earth

(B) unit extension of a predicate on the board in room 102.

The predicate B falls under its own scope, and this circularity is crucial to the generation of paradox.[3]

But set-theoretical paradox can arise in the absence of circularity and self-reference. Consider an infinite sequence of 1-place predicates E_1, E_2, … E_k, … , and let

S_k = {x| x=k or x is a non-empty finite extension of E_k or E_{k+1} or … E_{k+i} or … }. Now define the function ext* as follows:

ext*(E_k,E_{k+1},…) = S_k if at least one of E_k, E_{k+1},… has a non-empty finite extension;

otherwise, ext*(E_k,E_{k+1},…) = {Ø}, where Ø is the empty set.

For example, consider the following infinite sequence of 1-place predicates: 'integer between 1 and 5', 'natural number', 'NC Senator in 2003', '>0', '>1', … '>31', … .

Now, ext*(E_1,E_2,…E_k,…) = S_1 = {1, {2,3,4}, {Edwards, Dole}},

ext*(E_2,E_3,…E_k,…) = S_2 = {2, {Edwards, Dole}},

and ext*(E_4,E_5,…E_k,…) ={Ø}.

Now consider the following sequence of predicates:

P_1 member of ext*(P_2,P_3,…P_k,…)

P_2 member of ext*(P_3,P_4,…P_k,…)

.
.
.

P_k member of ext*(P_{k+1},P_{k+2},…)

.
.
.

We will show that this sequence of predicates generates a paradox.

<u>Proposition</u> $ext(P_k)=\{\emptyset\}$, for arbitrary k.

<u>Proof</u> Suppose towards a contradiction that $ext(P_k)\neq\{\emptyset\}$. Since $ext(P_k)=ext^*(P_{k+1},P_{k+2},\ldots)$, it follows that

(i) for some m>k, P_m has a non-empty finite extension.

We can also show

(ii) $ext(P_m) \neq\{\emptyset\}$.

For suppose towards a contradiction that $ext(P_m)=\{\emptyset\}$. Then, since $ext(P_m)= ext^*(P_{m+1},P_{m+2},\ldots)$, no predicate among P_{m+1}, P_{m+2}, … has a non-empty extension. But consider P_{m+1}, namely 'member of $ext^*(P_{m+2},P_{m+3},\ldots)$'. Since no predicate among P_{m+2}, P_{m+3},… has a non-empty finite extension, $ext^*(P_{m+2},P_{m+3},\ldots) = \{\emptyset\} = ext(P_{m+1})$. But then P_{m+1} has a non-empty finite extension. We have a contradiction, and this establishes (ii).

Given (i) and (ii), and since since $ext(P_m)= ext^*(P_{m+1},P_{m+2},\ldots)$, at least one of P_{m+1}, P_{m+2}, … has a non-empty finite extension. That is,

(iii) for some n>m, P_n has a non-empty finite extension.

We can also establish that

(iv) $ext(P_n) \neq\{\emptyset\}$,

by reasoning exactly similar to that which established (ii).

Given (iii) and (iv), at least one of P_{n+1}, P_{n+2}, … has a non-empty finite extension – say, the predicate P_q. By reasoning exactly similar to that which established (ii), we can show that $ext(P_q) \neq\{\emptyset\}$. And so we obtain that at least one of P_{q+1}, P_{q+2}, … - say, P_r - has a non-empty finite extension. Again we can show that $ext(P_r) \neq\{\emptyset\}$. And so on: the reasoning may be repeated indefinitely. So there are denumerably many predicates P_n, P_q, P_r, … with non-empty finite extensions other than $\{\emptyset\}$, where n, q, r, … >m. These extensions are all distinct – each contains a positive integer peculiar to it (for example, n is a member of ext(P_n) but not of ext(P_q) or ext(P_r) or …). So there are denumerably many of these extensions - and they are all members of ext(P_m). So P_m has an infinite extension, contradicting (i). This completes the proof of the proposition.

Since k is arbitrary, we have that for all k, ext(P_k)= {Ø}. In particular,

(a) ext(P_1)={Ø}.

Now P_1 is the predicate 'member of ext*(P_2,P_3,…P_k,…)', and all of P_2, P_3, … P_k, … have the same non-empty finite extension, namely {Ø}. So ext*(P_2,P_3,…P_k,…) = S_1 = {1, {Ø}}. So

(b) ext(P_1)={1,{Ø}}.

Since (a) and (b) yield a contradiction, we are landed in paradox.

Like the paradox of definability, this paradox does not display any self-reference: each predicate makes reference only to predicates further down the list. Again, an adequate treatment of this paradox must go beyond considerations of self-reference and circularity.[4]

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Bibliography

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Endnotes

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[1]. If we suppose that the phrase C does denote a number, say k, then we may infer that C denotes $\pi+6+k$, which is the sum of the numbers denoted by expression on the board in room 101. But then $k = \pi+6+k$, and we have a contradiction. So the phrase C does not denote a number, on pain of contradiction. But then the sum of the numbers denoted by expressions on the board is $\pi+6$ – and so C does denote a number, namely $\pi+6$. So we conclude that C does and does not denote a number – we are landed in paradox.

This paradox presented in Author 1994 and discussed further there. Even more tightly self-referential is the phrase discussed in Hilbert and Bernays 1939: 'the successor of the integer denoted by this phrase'.

[2]. There can be only one such expression. Suppose, towards a contradiction that D_m and D_n each denote p-1, where we may assume, without loss of generality, that m<n. Then D_m is given by:

D_m. $1+\max(D_{m+1},...D_n,...)$.

Since D_m denotes p-1, $\max(D_{m+1},...D_n,...)=p-2$. But since D_n denotes p-1, $\max(D_{m+1},...D_n,...)\geq p-1$. So $p-2\geq p-1$, and we have a contradiction.

[3]. Suppose first that B has a unit extension (an extension with just one member). Then, since there is just one moon of Earth and since B is itself a predicate on the board, ext(A) and ext(B) are both members of ext(B), and so B does not have a unit extension. Contradiction. Suppose second that B does not have a unit extension. Then ext(A) is the only member of ext(B) – so B

does have a unit extension.   Contradiction. Either way, we obtain a contradiction – we have a paradox.   This paradox is discussed further in Author 2000.

[4]. These paradoxes of definability and extension are companions to Yablo's version of the Liar, which is likewise free of circularity or self-reference. See Yablo 1993.