

Reference and paradox

Consider the phrase:

(B) the least integer not nameable in fewer than nineteen syllables.

There are only finitely many syllables of English. And so there are only finitely many phrases of English with fewer than nineteen syllables. But there are infinitely many integers, so there are integers not nameable by any phrase of English with fewer than nineteen syllables. And among these integers there is a least, call it k . Then the phrase B denotes the integer k . But the phrase B itself has eighteen syllables. So the number not nameable in fewer than nineteen syllables is denoted by a phrase with eighteen syllables. We have a contradiction, and we are landed in paradox.

This is Berry's paradox, presented by Berry in a letter to Russell in 1904. (Russell, by the way, says that the Berry phrase B denotes 111,777.)¹ Berry's paradox is one of the so-called *paradoxes of definability* - the others are Richard's paradox and Konig's paradox, both discovered in 1905, within a month of each other. It should be said that the label 'paradoxes of definability' is unfortunate. These paradoxes do not turn on the notion of definition in any ordinary sense of 'definition', and there's nothing particularly modal going on. Rather, the paradoxes turn on the notion of *reference* or *denotation*, the relation between a referring expression and its referent. This is one of the fundamental semantic relations, along with the relation between a predicate and its extension, and a sentence and its truth value. So like Russell's paradox about extensions and the Liar paradox about truth, the paradoxes of definability suggest that we do not have an

adequate understanding of our basic semantic concepts. Now, attention has been lavished on the Liar and Russell's paradox, while the paradoxes of definability have largely taken a back seat. So in this paper I'd like to redress the balance a little. The semantic notion of reference or denotation is no less fundamental than that of truth - indeed, arguably it's more fundamental - and we should attend to the paradoxes of denotation just as urgently as we attend to the Liar.

I. A simple paradox of denotation

It will help if we work with a simple paradox of denotation, much simpler than the Berry. The story goes as follows. Suppose I've just passed by a colleague's office, and I see denoting phrases on the board there. That puts me in the mood to write denoting phrases of my own, and so I enter an adjacent room, and write on the board the following expressions:

pi

six

the sum of the numbers denoted by expressions on the board in room 213.

Now I am in fact in room 213, though I believe that room 213 is my colleague's office. I set you the task of providing the denotations of these expressions.

You respond as follows:

"But we're in room 213! It's clear what the first two phrases denote. But what about the third? Let's call your third phrase P. Suppose P denotes the number k. Now, P denotes k if and only if the sum of the numbers denoted by expressions on the board in room 213 is k. So it follows that the sum of the numbers denoted by expressions on the board in room 213 is k. But then $k = \pi + 6 + k$, which is absurd. So P is pathological - it appears to denote a number but it doesn't, on pain of contradiction."

You continue:

"And now, since P does not denote a number, the only expressions on the board that do denote numbers are the first two. But then the sum of the numbers denoted by

expressions on the board in room 213 is $\pi+6$.”

You conclude:

“In the previous sentence, there is a token of the same type as P, call it P*. And P* does denote a number, namely $\pi+6$.”

There's something remarkable about this discourse. You establish that P is pathological, but the reasoning does not stop there. You reason past pathology. By reasoning that is natural and intuitively valid, you reach a phrase - P* - composed of the very same words as P, with the very same meaning. And yet, while P fails to denote, P* succeeds. We can *repeat* the words of a pathological denoting phrase and successfully denote. This phenomenon calls for explanation.

There is another way you could have continued. Instead of *repeating* P, you could *revisit* P:

"And now if P does not denote a number, then the only expressions on the board that do denote numbers are the first two. But now attend to the words on the board that make up the phrase P. They make reference to the phrases on the board that denote numbers, and the only phrases that do so are the first two. And the sum of those numbers is $\pi+6$.”

And now you conclude:

“So P denotes $\pi+6$.”

Again, something remarkable happens: you reason past the pathology of P, and find, by natural and intuitive reasoning, that P does denote. How can one and the same phrase be pathological and yet successfully refer? Again, an explanation is called for. (In their neglected writings on the definability paradoxes, Richard, Peano and Poincare show sensitivity to the fact that we can repeat and revisit pathological phrases.)²

So we want to explain these two stretches of discourse, one in which we *repeat* P, and the second in which we *revisit* P. In each case, your reasoning is natural and intuitive, and appears to

be valid. We should not block it by artificial, *ad hoc* means. Since the reasoning is natural, I would rather regard it as data that expresses semantic intuitions we have about the notion of denotation or reference. My strategy will be to find a plausible analysis that preserves the validity of the reasoning, and respects the data. And, I want to suggest, the explanation must be a *pragmatic* one, at least in the sense that the notion of *context* will play a crucial role. Given P and P*, two tokens of the same type, how can one fail to refer and the other succeed? Or given the phrase P, how can its semantic status change in the course of your reasoning? A natural response is this: *because of some change in the context*. So I shall be claiming that there are context-changes within our discourses. And I shall also be claiming that the term 'denotes' is sensitive to those changes. But let us first consider context-change and discourse analysis more generally.

II. Context-change and discourse analysis

We are all familiar with indexical terms, like 'I', 'now', 'here' and so on. It's a familiar idea that *context acts on content*. But it is increasingly being recognized that this is not a one-way street. The reverse direction holds as well: *content acts on context*. Stalnaker writes:

"context constrains content in systematic ways. But also, the fact that a certain sentence is uttered, and a certain proposition expressed, may in turn constrain or alter the context. ... There is thus a two-way interaction between contexts of utterance and contents of utterances."³

Or as Isard puts it:

"communications do not merely depend on the context for their interpretation, they change that context."⁴

At a given point in a conversation or discourse, the context will in part depend on what has been

said before. For example, the context may change as new information is added by the participants in the conversation. Over the last twenty years or so, the kinematics of context-change has been studied by philosophers, linguists and semanticists alike – for example, by Stalnaker, Lewis, Heim, Kamp, Reinhart, Grosz and Sidner, and many others.⁵

According to Stalnaker, the connection between context and available information is very tight indeed. Stalnaker says that a context should be *identified* with the body of information that is presumed to be available to the participants in the speech situation - the 'common ground', to use a phrase from Grice, or the shared presuppositions of the participants. As new utterances are produced, and new information is made available, the context changes. To keep track of these context changes, then, we need a running record of the information provided in the course of the conversation.

Lewis works with a related notion of the *conversational score*. The analogy is with a baseball score. A baseball score for Lewis is composed of a set of 7 numbers that indicate, for a given stage of the game, how many runs each team has, which half of which innings we're in, and the number of strikes, balls and outs. Notice that correct play depends on the score - what is correct play after two strikes differs from what is correct play after three strikes. Similarly for conversations: the correctness of utterances (their truth, or their acceptability in some other respect) depends on the conversational score.

Lewis identifies a number of components of a conversational score. One is the set of shared presuppositions of the participants, the common ground. Here Lewis follows Stalnaker. As Lewis puts it: "Presuppositions can be created or destroyed in the course of a conversation".⁶ As the set of presuppositions changes, the conversational score changes. Of course, the notion of

conversational score is a vivid way of capturing the notion of context. A change in the set of presuppositions is a change of context.

Another component of conversational score, according to Lewis, is the *standard of precision* that is in force at a given stage of the discourse. Suppose I say 'France is hexagonal'. If you have just said 'Italy is boot-shaped', and got away with it, then your utterance is true enough. The standards of precision are sufficiently relaxed. But if you have just denied that Italy is boot-shaped, and carefully pointed out the differences, then my utterance is far from true enough - the standards of precision are too exacting. The acceptability of what I say here depends on the conversational score, on the context, which is determined by what has been said before.

In both Stalnaker and Lewis we find the idea of tracking context-change by keeping a running record of shifts in the information presumed to be available to the conversants. A number of linguists have also developed this idea. For example, in Irene Heim's *file change semantics*, a 'file' contains all the information that has been conveyed up to that point - and the file is continually updated as the discourse moves on. The same basic idea informs Grosz and Sidner's theory of discourse structure and their notion of a *focus space*, and also Tanya Reinhart's analysis of sentence topics. The distinction in linguistics between given information and new information is also relevant here. For example, Chafe characterizes given information as the "knowledge which the speaker assumes to be in the consciousness of the addressee at the time of the utterance", and new information as "what the speaker assumes he is introducing into the addressee's consciousness by what he says."⁷ Context-change will be marked by the introduction of new information.

III. Context-change and the denotation discourse

Let's go back to the denotation discourses. Consider first the *repeating* discourse. It is natural to divide the discourse into four segments: first, where I produce the tokens on the board; second, where you argue to the conclusion that P is pathological and does not denote; third, where you build on that conclusion and produce P*; and fourth, where you reflect on P* and conclude that it does denote. We can identify changes in the common ground as this discourse progresses. For example, there is a change in our shared presuppositions as we move from the first segment of the discourse to the second – your assertion that we are in room 213 changes the context - it provides new information and a change in the common ground.

There is also a crucial difference between the first and second segments on the one hand, and the third and fourth on the other. The culmination of the reasoning of the second segment is the proposition that P is pathological and does not denote. This proposition is part of the common ground for the third and fourth segments. Throughout these subsequent segments, we are presupposing that P is pathological and does not denote. So in the transition from the second segment to the third, there is a shift in the context - there is a change in the body of information that is presumed to be available to us.

It is this context-change that I want to focus on - the one that occurs as we move from the second segment to the third and fourth. I shall call the new context *reflective with respect to P*. In general, a context associated with a given stage of a discourse is *reflective with respect to a given expression* if at that stage it is part of the common ground that the expression is semantically pathological (and so does not denote - or isn't true, or does not have an extension). So as we move from the second segment to the third and fourth, there is a change in the

conversational score - a shift from an unreflective context to a reflective one. We see exactly the same shift in the case of the revisiting discourse. There too we shift to a context that is reflective with respect to P.

IV. The action of context on content

Thus far, we have seen that content acts on context - newly available information changes the context. But there is a two-way interaction between context and content - context also acts on content. So now we want to see how the changes in context affect content.

Remember the challenge posed by the repeating discourse: P and P* are composed of the very same words with the very same meaning, and yet one fails to denote while the other denotes a definite number ($n+6$). Somehow the change in context produces this phenomenon, and our task is to explain how. Now if the context acts on content, we would expect at least one expression in the discourse to be sensitive in some way to context-change. I want to claim that 'denotes' is that expression. (When we examine the terms that appear in the denotation discourse, it's very hard to see a viable alternative.) So let us now see if this claim can be made out, not by any *ad hoc* maneuvering, but by a reasonable methodology. The methodology is this: given that the denotation discourse exhibits valid reasoning to a true conclusion, we are to find the most plausible analysis that preserves the validity of the reasoning and the truth of the conclusion.

How does the term 'denotes' behave in our discourse? Does its extension change at any point because of a change in context? Let i be the initial context associated with the first segment, where I produce the token P. My first use of 'denotes' is a component of the utterance P, and so occurs in the context i . So let us represent this first use of 'denotes' by 'denotes; _{i} '. This

representation does not commit us to the claim that 'denotes' is context-sensitive - we are only marking the fact that this use of denotes occurs in context i . Let us continue to attach the subscript ' i ' to each subsequent use of 'denotes' if no change of extension is forced upon us. If context has no effect on the extension of 'denotes' - if 'denotes' is a predicate constant - then the continued appearance of the subscript i will indicate no more than this: every use of 'denotes' is coextensive with every other. If on the other hand 'denotes' is context-sensitive, then the subscript i will reappear only if, for some reason or other, subsequent uses of 'denotes' inherit the extension that the context i determined for my first use of 'denotes'.

So at the outset of the discourse we have:

π

six

the sum of the numbers denoted $_i$ by expressions on the board in room 213.

Now, in order to determine the denotation of P , we have to determine what the expressions on the board denote $_i$. So the subscript i will continue to appear in the representation of your reasoning:

"It's clear what the first two phrases denote $_i$. But what about the third? Let's call your third phrase P . Suppose P denotes $_i$ the number k . Now, P denotes $_i$ k if and only if the sum of the numbers denoted $_i$ by expressions on the board in room 213 is k . So it follows that the sum of the numbers denoted $_i$ by expressions on the board in room 213 is k . But then $k = \pi + 6 + k$, which is absurd. So P is pathological - it appears to denote $_i$ a number but it doesn't, on pain of contradiction."

Nothing so far forces a different extension on 'denotes'; quite the reverse, in fact.

Let us pause here. As you conduct your reasoning, something is operating in the background - what we may call a *denotation-schema*. A denotation-schema is an exact analogue

of the perhaps more familiar truth-schema. An instance of the truth schema is:

'snow is white' is true if and only if snow is white.

An instance of the denotation schema is:

'3²' denotes 9 if and only if 3²=9.

The instance of the schema you have used is this:

P denotes_i k if and only if the sum of the numbers denoted_i by expressions on the board in room 213 is k.

Call this the i-schema. When you assess P by the i-schema, a contradiction results.

Let us return to the discourse. You continue:

"And now, since P does not denote_i a number, the only expressions on the board that do denote_i numbers are the first two. But then the sum of the numbers denoted_i by expressions on the board in room 213 is $\pi+6$."

It's clear that the first two occurrences of 'denotes' receive the subscript i, since they rely on your result that P does not denote_i. And for just the same reason, the occurrence of 'denote' in the token P* receives the subscript i as well. The occurrences of 'denotes' in this present stage of the discourse inherit their contextual subscripts from the previous stage. Notice that in a strict sense, P* is a repetition of P - it is composed of the very same words, with the very same meaning, and the very same extensions.

Now let's move to your final use of 'denotes'. You continue:

"In the previous sentence, there is a token of the same type as P, call it P*. And P* does denote a number, namely $\pi+6$."

Should we attach the subscript i to this final use of 'denotes'? The answer is no. You have reached a true conclusion here through valid reasoning. According to your assessment here, P* denotes. But P* does not denote_i - plug P* into the i-schema, and you'll get a contradiction, just

as you did with P. So here, a shift in extension *is* forced upon us - P^* denotes_r, let us say, where 'denotes_i' and 'denotes_r' have different extensions. P^* is in the extension of 'denotes_r', but it is not in the extension of 'denotes_i'.

What produces this shift in extension? The change in context - specifically, the shift to a context which is reflective with respect to P. When you assess P^* , and declare that it denotes, you assess it in a context in which it is part of the common ground that P is pathological. The schema by which you assess P^* - the r-schema - provides an assessment of P^* in the light of P's pathologicity. The relevant instance of the r-schema is this:

P^* denotes_r k if and only if the sum of the numbers denoted_i by expressions on the board in room 213 is k.

The right hand side of the biconditional is true for $k=\pi+6$, given that P is pathological, and doesn't denote_i. And so it follows that P^* denotes_r $\pi+6$, *as you say*.

In a nutshell, then, we explain your different assessments of P and P^* this way: you assess P by the unreflective i-schema, and you assess P^* by the reflective r-schema. There is no intrinsic difference between P and P^* - the difference lies in the schema by which they are evaluated.

Notice that P also denotes_r $\pi+6$, just like P^* . (Plug P into the r-schema, and that's the result you'll get.) So P fails to denote, and P does denote. But there is no contradiction here: P fails to denote_i, but it does denote_r. Compare: sometimes an utterance of 'France is hexagonal' is true enough, and sometimes it isn't. It depends on the conversational score, in particular on the standards of precision that are in force. In a loosely analogous way, whether or not P denotes depends on the standard of assessment: do we apply the unreflective i-schema or the reflective r-

schema?

The revisiting discourse is analysed in just the same way. When we revisit P, we do so in a context that is reflective with respect to P. Having concluded that P does not denote_i, and is pathological, we re-assess P by the r-schema and find that P does denote_r. In both discourses, the extension of 'denotes' undergoes a change. 'Denotes' is a context-sensitive term that may shift its extension according to context.

V. Singularities

We've seen that P is excluded from the extension of 'denotes_i', but not from the extension of 'denotes_r'. What else is excluded from the extension of 'denotes_i'? And what is the relation between the extension of 'denotes_i' and the extension of 'denotes_r'?

A possible response here appeals to a hierarchy. The claim might be that when we move from an unreflective context to a reflective one, we move to an essentially richer language. The predicate 'denotes_i' is the denotation predicate of the unreflective context; the predicate 'denotes_r' is the more comprehensive denotation predicate of the semantically richer language of the reflective context. On such a hierarchical account, the extension of 'denotes_i' is properly contained in the extension of 'denotes_r'.

As far as there is an orthodoxy regarding the definability paradoxes, some kind of hierarchical account is it. (For example, van Heijenoort writes: "'Today [Richard's] paradox is generally considered solved by the distinction of language levels".⁸) But I think we should resist the hierarchy. There are, I would argue, a number of problems with the hierarchical account.⁹ Let me just mention here the most blatant: the hierarchical account offers too regimented an

account of natural language. Surely English does not contain infinitely many distinct denotation predicates, but just one. And surely the stratification of English into a hierarchy of distinct languages is highly artificial. With Tarski, we may doubt

"whether the language of everyday life, after being 'rationalized' in this way, would still preserve its naturalness and whether it would not rather take on the characteristic features of the formalized languages."¹⁰

My proposal is in a strong sense *anti-hierarchical*. The leading idea is that in pathological denoting phrases like P, there are *minimal* restrictions on occurrences of 'denotes'. At this point, a pragmatic principle of interpretation comes into play: the principle of Minimality. According to Minimality, restrictions on occurrences of 'denotes' are kept to a minimum: we are to restrict the application of 'denotes' only when there is reason to do so.

Suppose you innocently say:

"The phrase 'the square of 2' denotes 4".

Here, your use of 'denotes' is quite unproblematic. Should P be excluded from its extension? Minimality says no - because there is no need to exclude it. We have seen that in a suitably reflective context, P does denote a number (specifically, P denotes_r π+6). Now the context of your utterance here is quite unconnected to my production of P. But Minimality says that your unconnected context should be treated *as if it were reflective with respect to P*. In your unconnected, neutral context of utterance, P can be counted among the phrases that denote - and so by Minimality, it is so counted. If there's no need to exclude P, then include it.

If we adopt Minimality we respect a basic intuition about predicates. Intuitively, we take a predicate to apply to everything with the property that the predicate picks out. In general, if an individual has the property picked out by the predicate φ, then that individual is in the extension

of φ . The more restrictions we place on occurrences of 'denotes', the more we are at odds with this intuition. We do expect any solution to a genuine paradox to require some revision of our intuitions. But the more a solution conflicts with our intuitions, the less plausible that solution will be.

For example, the hierarchical stratification of the denotation predicate involves massive restrictions on occurrences of 'denotes'. On a standard Tarskian line, the referring expression 'the square of 2' is of level 0; the denoting phrase "the number denoted by 'the square of 2'" is of level 1; and so on, through the levels. A use of 'denotes' in an utterance of level 1 has in its extension all referring expressions of level 0, *and no others*. So all sentences of level 1 and beyond are excluded from the extension of such a use of 'denotes'. Godel remarks of Russell's type theory that "...each concept is significant only ... for an infinitely small portion of all objects."¹¹ A similar complaint can be made about a standard hierarchical account of denotation: an ordinary use of 'denotes' will apply to only a fraction of all the denoting expressions. Minimality keeps surprise to a minimum: our uses of 'denotes' apply to almost all denoting phrases. We are sometimes forced to restrict 'denotes' - we must, for example, limit the extension of 'denotes_i' by excluding P. Still, according to Minimality, we exclude only those denoting expressions that cannot be included.

So my proposal identifies what I call *singularities* of the concept *denotes*. For example, P is a singularity of 'denotes_i'. Notice that P is a singularity only in a context-relative way - P is not a singularity of 'denotes_r' or your neutral use of 'denotes'. So in my view we should not stratify the concept of denotation; rather we should identify its singularities. Godel once made the following tantalizing remark:

“It might even turn out that it is possible to assume every concept to be significant everywhere except for certain 'singular points' or 'limiting points', so that the paradoxes would appear as something analogous to dividing by zero. Such a system would be most satisfying in the following respect: our logical intuitions would then remain correct up to certain minor revisions, i.e. they could then be considered to give an essentially correct, only somewhat 'blurred', picture of the real state of affairs.”¹²

I take my singularity proposal to be very much in the spirit of Godel's remarks. We retain a single denotation predicate which undergoes minimal changes in its extension according to context. There is no wholesale revision of the notion of denotation; no division of 'denotes' into infinitely many distinct predicates; no splitting of everyday English into an infinite hierarchy of languages.

VI. A glimpse of the formal theory

The main task of the formal theory is to identify the singularities of a given occurrence of 'denotes'.¹³ And this is carried out by way of certain kinds of trees - denotation trees. We will come to those in a moment.

But first consider P again. Suppose we represented the token P as an ordered pair $\langle \text{type}(P), i \rangle$, where the first element is the type of P, and the second indicates the appropriate representation of 'denotes' in P (viz., 'denotes_i'). Then something is missing. This representation does not distinguish P from P*, yet the former denoting expression is pathological and the latter isn't. There is something more to consider: the schema by which the token is given denotation conditions. In the second segment of the discourse, P is evaluated by the i-schema; in the third segment, P* is evaluated by the r-schema, a schema that is reflective with respect to P. So we capture the denotation discourse more perspicuously if we represent P by the ordered *triple* $\langle \text{type}(P), i, i \rangle$, where the third element indicates that the schema by which P is assessed is the i-

schema, and P^* by the triple $\langle \text{type}(P), i, r \rangle$, indicating that P^* is assessed by the r -schema.

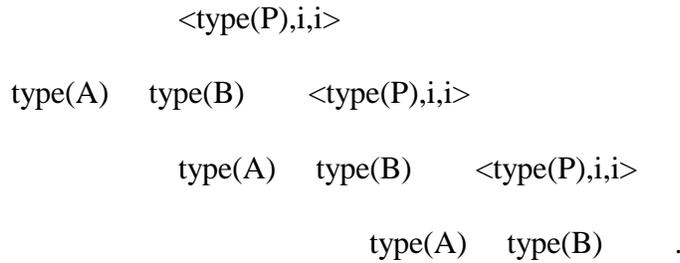
We may of course evaluate P and P^* by other schemas. But it is the evaluation of P by the i -schema – the schema associated with P 's context of utterance - that leads to the conclusion that P is pathological. And it is the evaluation of P^* by the r -schema – the schema associated with P^* 's context of utterance - that leads to the conclusion that P^* has a determinate referent. So if we are after an analysis of the denotation discourse, the representation $\langle \text{type}(P), i, i \rangle$ of P is privileged over other representations of P , and $\langle \text{type}(P), i, r \rangle$ is likewise a privileged representation of P^* . We will call these representations the *primary representations* of P and P^* .

Let's now construct what we will call the *primary denotation tree* for P . The top node is the primary representation of P : $\langle \text{type}(P), i, i \rangle$. The third element of the triple indicates that we are after the denotation _{i} of P . Now P makes reference to certain denoting expressions - the ones on the board. To determine a denotation _{i} for P , we must first determine the denotation of these expressions – more specifically, we are after what these expressions denote _{i} , since the occurrence of 'denotes' in P is given by 'denotes _{i} ', as the second element of the primary representation indicates. We can draw a diagram to capture this:

$$\begin{array}{c} \langle \text{type}(P), i, i \rangle \\ \text{type}(A) \quad \text{type}(B) \quad \langle \text{type}(P), i, i \rangle \end{array}$$

Here A and B stand for the first two expressions on the board. Since they contain no context-sensitive terms, we represent them via their types. The representation of P at the second tier reflects the fact that we are after what P denotes _{i} - so the third member of the triple is i . Now the primary representation of P appears again at the second tier. And to determine the denotation _{i} of P , we must again determine the denotations of the expressions on the board. And so we are

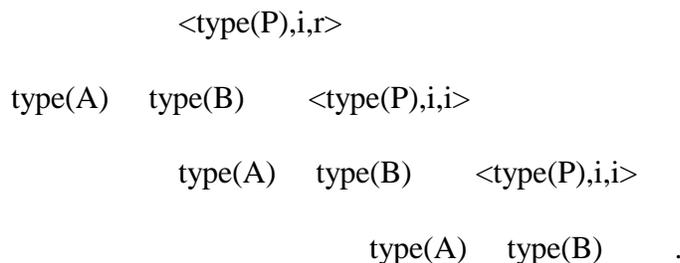
led to a third tier, and a fourth, and so on:



No branches extend from A and B, since they do not make any references to other denoting phrases. But P is caught in a circle, and the tree extends indefinitely. This is the primary denotation tree for P. This tree has an infinite branch, and P repeats on this branch. The repetition indicates that P is pathological, and a singularity of 'denotes!'. The attempt to determine a denotation_i for P breaks down.

(The treatment of the Berry phrase follows the same general pattern. The primary denotation tree for the Berry phrase contains an infinite branch, indicating the pathology of the Berry phrase. The Berry phrase is identified as a singularity of the occurrence of 'denotes' or 'nameable' in the phrase itself.)

While P is identified as pathological, it is a different story with P*. The primary denotation tree for P* looks like this:



At the top is the primary representation of P^* . The tree reflects the way in which we determine a denotation_r for P^* . The occurrence of ‘denotes’ in P^* is given by ‘denotes_i’. So to determine what P^* denotes_r, we must first determine the denotation_i of the phrases to which P^* makes reference – and this is indicated by the second tier of the tree. Now the tree has an infinite branch too, but notice that the primary representation of P^* does *not* repeat on this branch. The primary representation of P does repeat, but the primary representation of P^* does not. In the formal account, this indicates that while P is pathological, P^* is not. Intuitively, P^* stands above the circle in which P is caught.

Formally, we can take things a little further by *pruning* this tree. We prune the tree by terminating any infinite branch at the first occurrence of a non-repeating node. Here is the pruned tree:

$$\begin{array}{c} \langle \text{type}(P), i, r \rangle \\ \text{type}(A) \quad \text{type}(B) \end{array}$$

The pruned tree indicates that the denotation_r of P^* depends only on the denotation of A and B . And that's just what we found: the denotation_r of P^* is $\pi+6$.

Here then is a glimpse of the formal treatment of the *repeating discourse*. The treatment of the *revisiting discourse* should be obvious. When you revisit P at the third stage of the discourse, you bring to bear the reflective r -schema. So at the third stage of the revisiting discourse, we should represent P by the triple $\langle \text{type}(P), i, r \rangle$, indicating that we are after the denotation_r of P . This is not the primary representation of P , but what we will call a *secondary representation* of P , since the third element stands for a context other than P 's own context of

utterance. Starting out with this secondary representation, we can construct the corresponding secondary denotation tree for P. It is easy to see that this secondary tree for P is identical to the primary tree for P*. And as before this tree may be pruned, indicating that P, like P*, denotes, $\pi+6$.

VII. The object language and the language of the theory

Here, then, is a glimpse of the formal theory. Let me now make some more general remarks about it. The theory is a theory about a natural language, English, with particular focus on the term 'denotes'. So the object language is a portion of English including the term 'denotes'. The singularity theory is a theory *of* a context-sensitive term, but no context-sensitive term will appear *in* the language of the theory. So there is a separation between the object language and the language of the theory. So we should ask: what is the relation between the two?

For some theories of paradox, the relation is that of *object language* to *metalanguage*. Consider, for example, Kripke's theory of truth (in Kripke 1975). The object language is a certain fragment of English, containing its own truth predicate. This language can express its own concept of truth, in the sense that it contains a predicate whose extension comprises exactly the true sentences of the language. But there are certain ordinary semantic concepts that the object language cannot express – for example, *untrue*. Now this concept can be expressed in the semantically richer language of the theory. The language of the theory is, in Tarski's sense, a metalanguage for the object language. The object language is fully translatable into the language of the theory, and the language of the theory can say more besides.

This shows the limitations of Kripke's theory. If the theory is a theory of an object language that cannot express an ordinary semantic concept like 'untrue', then we do not really

have a theory of truth for *English*. Natural languages, says Tarski, are *universal*: "if we can speak meaningfully about anything at all, we can also speak about it in colloquial language"¹⁴.

In particular, says Tarski, natural languages are *semantically universal*, in the sense that they can express all of their own semantic concepts – not just truth, but untruth as well. The need to ascend to a hierarchy to express ordinary semantic concepts shows that the scope of Kripke's theory is restricted. I would also argue that the same kind of ascent is forced upon other theories of truth, such as the revision theory, or those of Feferman and McGee. When the language of the theory is a metalanguage, the scope of the theory is restricted.¹⁵

In contrast, the language of the singularity theory is *not* a metalanguage for the intended object language – namely, the fragment of English containing 'denotes'. The singularity theory does not attempt to provide a model of this target language. Rather, its job is to identify singularities, by describing the semantic and pragmatic behaviour of the denotation predicate. Here, the object language *cannot* be translated into the language of the theory: the object language is too rich for that. For one thing, 'denotes' is a context-sensitive term, and the theory contains no context-sensitive term. For another, the scope of any use of 'denotes' is far too wide to be captured by any term of the theory. An ordinary use of 'denotes' applies to *every* denoting phrase which is not identified as a singularity. And that includes all such denoting phrases in the language of the theory itself, and indeed in all other languages, actual and possible.¹⁶ Since the scope of each use of 'denotes' is as close to global as it can be, it seems to me that the singularity theory goes a long way to accommodate Tarski's intuitions about semantic universality.

VIII. Consequences for deflationism

I turn now to the consequences for one kind of deflationism about reference – the disquotational account. According to a disquotational account of denotation or reference, there is no more to denotation than is given by the instances of the following disquotational schema:

‘e’ denotes n if and only $e=n$,

where ‘e’ and ‘n’ are denoting expressions. This leads naturally to the following disquotational definition of denotation for a language L:

x denotes n iff $x='e_1' \ \& \ e_1=n$
or $x='e_2' \ \& \ e_2=n$
or ...
or $x='e_k' \ \& \ e_k=n$
or ... ,

where ‘e₁’, ‘e₂’, ... ‘e_k’, ... are the denoting expressions of L.

Suppose we try to maintain both disquotationalism and the singularity account. In particular, consider the disquotational definition for ‘denotes_r’. If we put $x=P$ and $n=\pi+6$, we obtain:

P denotes_r $\pi+6$ if and only if the sum of the numbers denoted_i by expressions on the board is $\pi+6$.

This is a true biconditional. But what about the occurrence of ‘denotes_i’ on the right? *This cannot be treated disquotationally.* If we plug P into the disquotational definition of ‘denotes_i’, and suppose that P denotes_i some number k, we reach the absurdity that $k=\pi+6+k$. So we cannot provide a disquotational account of the occurrence of ‘denotes_i’ in P. If we accept the singularity account, we cannot accept disquotationalism. There are genuine denoting expressions containing uses of ‘denotes’ which the disquotationalist cannot handle - in particular, the use of

'denotes' in P cannot be treated disquotationally. And P cannot be dismissed as merely pathological – it is a genuine denoting phrase. In short, denotation cannot be disquoted away.

IX. A new paradox of definability

Finally, I'd like to turn to a new paradox of definability and see how the singularity theory handles it. It is a standard idea that the paradoxes of definability turn on self-reference – the phrase P makes reference to itself, and the Berry phrase includes itself in its scope. Every definability paradox with which I am familiar involves self-reference. But I think we can formulate paradoxes of definability without self-reference.

Let E_1, E_2, \dots be a list of denoting expressions - it may be finite or infinite. The list might be '4²', 'the successor of 2', 'England'; or it might be 'one', 'three', 'five', ... 'thirty one', As our examples show, some of these expression may denote positive integers (i.e. 1 or 2 or 3 or ...). Let the expression ' $\max(E_1, E_2, \dots)$ ' denote the largest positive integer denoted by an expression on the list. If denumerably many distinct positive integers are denoted by expressions on the list (so that there is no largest positive integer denoted), ' $\max(E_1, E_2, \dots)$ ' denotes w , the first infinite ordinal; and if no expression on the list denotes a positive integer, ' $\max(E_1, E_2, \dots)$ ' denotes 0. So, for example,

$$\max('4^2', 'the successor of 2', 'England') = 16,$$

$$\max('London', 'Chapel Hill', 'Tel Aviv') = 0, \text{ and}$$

$$\max('one', 'three', 'five', \dots 'thirty one', \dots) = w.$$

Now consider the following infinite list of denoting expressions:

$D_1. 1+\max(D_2,D_3, \dots , D_n, \dots).$

$D_2. 1+\max(D_3, D_4, \dots , D_n, \dots).$

.

.

$D_n. 1+\max(D_{n+1},\dots,D_{n+i},\dots).$

$D_{n+1}.1+\max(D_{n+2},\dots,D_{n+i},\dots).$

.

.

Notice that there's no self-reference or circularity here. Each phrase makes reference only to phrases further down the list. Suppose towards a contradiction that for some arbitrary n , D_n denotes a positive integer, say p . Now D_n is given by:

$D_n. 1+\max(D_{n+1},\dots, D_{n+i},\dots).$

Since D_n denotes p , $\max(D_{n+1},\dots, D_{n+i}, \dots) = p-1$. So there is an expression D_k among $D_{n+1}, \dots, D_{n+i}, \dots$ which denotes $p-1$.¹⁷ D_k is given by:

$D_k. 1+\max(D_{k+1},\dots, D_{k+i},\dots).$

Since D_k denotes $p-1$, $\max(D_{k+1},\dots, D_{k+i},\dots) = p-2$. So there is an expression D_l among $D_{k+1}, \dots, D_{k+i}, \dots$ which denotes $p-2$. And so on. Continuing in this way (for $p-3$ more steps), we obtain an expression D_z which denotes 1. D_z is given by:

$D_z. 1+\max(D_{z+1},\dots, D_{z+i},\dots).$

Since D_z denotes 1, $\max(D_{z+1},\dots, D_{z+i},\dots) = 0$. By the definition of 'max', none of $D_{z+1}, \dots, D_{z+i}, \dots$ denote a positive integer. In particular,

(i) D_{z+1} does not denote a positive integer.

Now D_{z+1} is given by:

$$D_{z+1}. 1+\max(D_{z+2},\dots,D_{z+i},\dots).$$

Since none of $D_{z+2}, \dots, D_{z+i}, \dots$ denote a positive integer, $\max(D_{z+2},\dots,D_{z+i},\dots) = 0$. But then D_{z+1} denotes $1+0$. That is,

(ii) D_{z+1} denotes a positive integer, namely 1.

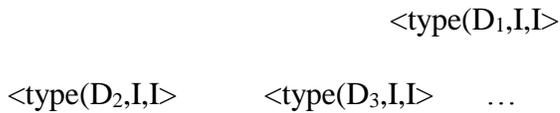
From (i) and (ii), we obtain a contradiction.

By our reductio argument, we have shown that no D_n denotes a positive integer, for any n . Every D_n is pathological. So for all n , $\max(D_n,\dots,D_{n+i},\dots) = 0$. In particular, then, $\max(D_2,\dots,D_n,\dots) = 0$; $\max(D_3,\dots,D_n,\dots) = 0$; and so on. But now reconsider D_1 . Since $1+\max(D_2,\dots,D_n,\dots) = 1$, D_1 denotes 1. Similarly, D_2 denotes 1, and in general D_n denotes 1. To sum up: no D_n denotes a positive integer, and every D_n denotes a positive integer (namely 1). We are landed in paradox.¹⁸

This paradox of definability is obviously more complicated than the simple one generated by P on the board. But the treatment of it is essentially the same. The discourse here has the same overall structure: first, denoting phrases are displayed; second, we reason to the conclusion that they are pathological and fail to denote; and third, we reason past pathology, and find that the phrases do denote. When we first assess D_1, D_2 , and the rest as pathological, we assess them by an unreflective denotation schema, analogous to the i -schema in the case of P . We find that all the D_n 's are pathological, and fail to denote. And with the availability of this new information, the common ground shifts, and we move to a reflective context. We now assess $D_1, D_2 \dots$ by a new reflective schema, analogous to the r -schema. Just as we reason first that P does not denote and then conclude that it does, so we find first that D_1 , and all the others, fail to denote, and then

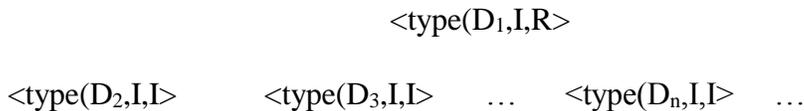
go on to conclude that they do. There is no contradiction here. Rather there is a shift in the denotation-schemas by which we assess $D_1, D_2 \dots$ triggered by the shift to a reflective context.

Let I be the initial context in which the D_n 's are first produced. And let R be the reflective context in which we assess the D_n 's on the basis of their pathology. Then the primary tree for D_1 looks like this:



This tree clearly contains infinite branches, and the formal theory will deliver the result that D_1 is pathological, and a singularity of denotes_I. And similarly for D_2, D_3, \dots etc.

In our reasoning, we revisit D_1 and re-assess it via the reflective R -schema. This is captured by the following secondary tree for D_1 :



Notice that the second tier is composed of primary representations of D_2, D_3, \dots , each of which head an unfounded primary tree. Setting aside the details of the formal theory, the idea is that this tree indicates that the denotation_R of D_1 is to be determined in the light of the pathology of D_2, D_3, \dots etc. This is just what we do at the reflective stage of our reasoning. Having

determined that D_2, D_3, \dots etc fail to denote, we calculate that $\max(D_2, D_3, \dots) = 0$, and hence that $1 + \max(D_2, D_3, \dots) = 1$, concluding that D_1 denotes 1.

X. Concluding remarks

I have focused here on denotation. But I think we can take the same kind of approach to the other fundamental semantic relations – between a predicate and its extension, and a sentence and its truth value. Parallel to our denotation discourse, there are discourses related to Russell's paradox and the Liar, and there too, I believe, we can provide a singularity account. But that, obviously, is a topic for another day.

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Footnotes

¹ Berry's paradox is reported in Russell 1908.

² See Peano 1906, Poincare 1906 and 1909, and Richard 1905 and 1907. For a discussion, see Simmons 1994.

³ Stalnaker 1975, in Stalnaker 1999, p.66.

⁴ Isard 1975.

⁵ I am grateful to Zoltan Szabo for emphasizing the relevance of Stalnaker's work to my account of the Liar. Glanzberg 2001 also stresses the relevance to the Liar of context-change and discourse analysis.

⁶ Lewis 1979, in Lewis 1983, p.233.

⁷ Chafe 1976, p.30.

⁸ See van Heijenoort 1967, p.142.

⁹ For more on this, see Simmons 1993.

¹⁰ Tarski 1983, p.267.

¹¹ Kurt Godel 1944, in Schilpp 1944, p. 149.

¹² *op. cit.*

¹³ Once we have a formal theory that identifies singularities of a given use of 'denotes', we can offer suitably restricted denotation schemas; for example,

If 'e' is not a singularity of 'denotes_i', then 'e' denotes_i n iff e=n,
where 'e' and 'n' are denoting expressions. (So in response to J.C. Beall's accompanying commentary, I endorse option 6.5).

¹⁴ Tarski 1983, p.164.

¹⁵ For more along these lines, see Simmons 1993, chapters 3-4.

¹⁶ Notice that if we regard the language of the theory as a classical, regimented language, it will

not contain its own denotation predicate – for that, we would need a richer metalanguage for the language of the theory. Thus there will be denoting phrases involving this denotation predicate that cannot be expressed in the language of the theory. But as long as these phrases are not identified as singularities, they will be in the extension of an ordinary use of ‘denotes’.

¹⁷ There can be only one such expression. Suppose, towards a contradiction, that D_m and D_n each denote $p-1$, where we may assume without loss of generality that $m < n$. Then D_m is given by:

$D_m. 1 + \max(D_{m+1}, \dots, D_n, \dots)$.

Since D_m denotes $p-1$, $\max(D_{m+1}, \dots, D_n, \dots) = p-2$. But since D_n denotes $p-1$, $\max(D_{m+1}, \dots, D_n, \dots) \geq p-1$. So $p-2 \geq p-1$, and we have a contradiction.

¹⁸ This paradox is a companion to Yablo’s paradox about truth – see Yablo 1993.