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Poincaré and Paradox

At the heart of Poincaré's treatment of the paradoxes is the notion of a *non-predicative definition*. I shall suggest that Poincaré characterizes this notion in two distinct ways - consequently, we can distinguish two rather different approaches that Poincaré took to the paradoxes. I shall argue that neither approach works; neither avoids the return of paradox. However, I go on to argue that the second approach hints at a new kind of solution to the paradoxes - a `contextual' solution. To fix ideas, I shall focus on one paradox to which Poincaré gives special attention: Richard's paradox of definability. But I believe that my discussion has broader scope, extending beyond the definability paradoxes.

I. Richard's paradox

Richard (1905) presents his paradox as follows. A certain set of numbers, the set E, is defined through the following considerations: write in alphabetical order all permutations of pairs of letters of the English alphabet, followed by all permutations of triples of letters of the alphabet taken in alphabetical order, and so on for quadruples, quintuples and so on. Cross out all permutations of letters that do not define numbers. E is the set whose members are u_1, the first number defined by a permutation, u_2, the second number defined by a permutation, and so on: that is, E is the set of all numbers that can be defined by finitely many words. Now, consider the following collection G of letters:

(G) Let p be the digit in the nth decimal place of the nth number of the set E; let us form a number having 0 for its integral part and, in its nth decimal place, p+1 if p is not 8 or 9, and 1 otherwise.

Call the number so defined N. Then N cannot belong to the set E. If it were the nth member in E, then the digit in its nth place would be the same as the digit in the nth decimal place of the nth number, which is not the case. Yet N is defined by finitely many words: hence it should belong to the set E. So we have a contradiction.[1]

II. Non-predicativity via vicious circles

According to Poincaré, we cannot define the set E in terms of E itself. The members of E must be defined independently of E. Now consider N. N is a member of E since it is defined by G, a finite collection of letters. But this definition *does* rely on E. And so there is a *vicious* *circle* in our definition of E. We must avoid this vicious circle by restricting the set E to those numbers that can be defined by a finite number of words *that do not mention E*. According to Poincaré, G does not define a number because the definition of a member of E cannot rely on E, on pain of a vicious circle (see Poincaré 1906, p. 307).

Poincaré extends the diagnosis to other paradoxes, in particular that of Burali-Forti. Here we consider the set Ord of all ordinal numbers, and the ordinal number α that corresponds to the order type of this set. We are quickly led to a contradiction, for α is a larger ordinal number than any in the set of *all* ordinals, and so larger than itself. Again Poincaré points to a vicious circle, this time in the definition of Ord: one of the members of Ord, namely α, is defined in terms of the set Ord itself. We must, then, introduce not an unrestricted set of all ordinals, but rather the set of all those ordinals that can be defined independently of the set itself (see Poincaré 1906, p. 307). An unrestricted definition of a set of all ordinals, like the unrestricted definition of Richard's set E of all numbers defined by finitely many words, contains a vicious circle. Such definitions Poincaré calls *non-predicative*. In general:

> ... the definitions that must be regarded as non-predicative are those that contain a vicious circle. (Poincaré 1906, p. 307)[2]

Suppose that we accept Poincaré's way out.[3] Suppose we agree with Poincaré that the collection G of letters does not define a number, because the definition of E would otherwise contain a vicious circle. Then there is an immediate difficulty. For, since G does not define a number, we have to cross it out. If we do cross out G, we *will* obtain an enumeration of sentences which define numbers.[4] And if E is the corresponding totally defined set of numbers, defined free of any vicious circle, then G acquires a meaning, and defines a number not in E. The problem for Poincaré is that if we restrict the set E to the set of numbers defined by a finite number of words that do not mention E, then we *can* define a new number by a finite expression that mentions E, without any vicious circle; Poincaré's restriction itself provides the conditions for lifting that restriction. G emerges as a perfectly meaningful sentence that does define a number. In short, it is no solution to reject G as meaningless, for that very rejection will render G meaningful.[5]

III. Non-predicativity via mutable classifications

If we take Poincaré's way out, we "cross out" the diagonal definition G, and continue on. But this yields a denotation for G, and so G has to be reinstated as the definition of a number. Now we might take the re-instatement of G as a second step in a process that repeatedly produces new numbers. Since G is a diagonal definition, it will define a new number, and we obtain an enlarged set of numbers definable by finite phrases. And by a further diagonal definition, we can in turn add to this set. The set of numbers defined by finitely many words will be constantly changing.

This process is incompatible with Poincaré's stated solution. The process is generated only if we take G to define a number - and according to Poincaré's way out, G does not define a number, on pain of a vicious circle. However, we can find in Poincaré's writings another characterization of non-predicative definitions - a characterization that is sensitive to this process.[6]

We have seen that Poincaré 1906 characterizes non-predicative definitions as those that contain a vicious circle. In a later work, Poincaré offers an *alternative* characterization of non-predicative definitions and classifications.[7] It goes like this:

> ... we draw a distinction between two types of classications applicable to the elements of infinite collections: the predicative classifications, which cannot be disordered by the introduction of new elements; the non-predicative classifications in which the introduction of new elements necessitates constant modification. (1909, p. 47)

Poincaré gives this illustration of a *predicative* classification:

> Let us suppose for example that we classify the integers into two families according to their size. We can recognize whether a number is greater or less than 10 without having to consider the relation of this number with the set of the other integers. Presumably, after the first 100 numbers have been defined, we shall know which among them are less than and which are greater than 10. When we then introduce the number 101, or any one of the numbers which follow, those among the first 100 integers which were less than 10 will remain less than 10, those which were greater will remain greater; the classification is predicative. (*ibid*)

Poincaré contrasts this with a *non-predicative* classification:

> In order to classify the integers, or the points in space, I shall consider the sentence which defines each integer or each point. Since it can happen that the same number or the same

> point can be defined by many sentences, I shall arrange these sentences in alphabetical order and I shall choose the first among these. With this as a condition, this sentence shall end with a vowel or with a consonant, and the classification can be made according to this criterion. But this classification would not be predicative; by the introduction of new integers, or of new points, sentences which had no meaning could acquire one. And then to the list of sentences which define an integer or a point already introduced, it will become necessary to add new sentences, which up to this point were devoid of meaning, have just acquired a meaning, and which define precisely this same point. It can happen that these new sentences assume the first position in the alphabetical order, and that they end with a vowel, whereas the previous sentences ended in a consonant. And then the integer or the point which had been provisionally placed in one category will have to be transferred to another. (1909, p. 48)

The situation is similar with Richard's paradox. Here, we classify the numbers into two groups: those that are definable by finitely many words, and those that are not. Given our list of sentences that define numbers, we obtain the members of E. But once we have E, then there are sentences, like G, that were devoid of meaning and have just acquired a meaning. And now the number N, first placed in the category of numbers that were not definable in finitely many words, is placed in the other category. We have now an expanded set of numbers that are definable by finitely many words of English. But then there will be sentences of English that make reference to this set (for example, this very sentence); and among these sentences there will be diagonal definitions that define a number not in this expanded set. Such sentences will lead to a further expansion of the set of numbers definable by finitely many words. So our classification requires constant modification. The classification is mutable, and it is this, says Poincaré, that is the source of Richard's antinomy, and of others too. Poincaré writes:

> Formal logic is nothing but the study of the properties common to all classifications; it teaches us that two soldiers who are members of the same regiment belong by this very fact to the same brigade, and consequently to the same division; and the whole theory of the syllogism is reduced to this. What is, then, the condition necessary for the rules of this logic to be valid? It is that the classification which is adopted be <u>immutable</u>. We learn that two soldiers are members of the same regiment, and we want to conclude that they are members of the same brigade; we have the right to do this provided that during the time spent carrying on our reasoning one of the two men has not been transferred from one regiment to another.
>
> The antinomies which have been revealed all arise from forgetting this very simple condition: a classification was relied on which was not immutable and which could not be so ... . (1909, p. 45)

Poincaré draws this moral: "Avoid non-predicative classifications and definitions" (1909, p. 63). In generating Richard's paradox, we rely on a non-predicative classification, between numbers that are finitely definable and those that are not. Following Poincaré, we must eschew this classification, and cross out the phrase G, since that phrase relies on the classification. Here we have a new reason to cross out G.

But still the old difficulty remains. Our list will now be composed of the unproblematic definitions of numbers, those that rely on predicative classifications only. And now the classification between numbers that are finitely definable and those that are not is rendered predicative: we can place each number in one or the other category, according to whether it is defined by a phrase in our list. So we must re-instate G - and the paradox returns. We cannot take Poincaré's way out.

IV. A contextual approach to paradox

But perhaps we can take another way. Recognizing that the classification requires modifications, *we might let the modifications happen*. We might allow that in the course of our reasoning, an expression that had no meaning acquires one. Let me illustrate what I have in mind by way of a simple paradox of definability.[8] Then I shall return to Richard's paradox.

Suppose that at noon 7/1/95 I write on the board in room 101 the following expressions:

A. the ratio between the circumference and diameter of a circle.

B. the positive square root of 36.

C. the sum of the numbers defined by expressions on the board in room 101 at noon 7/1/95.

I believe that room 101 is the room next door, and that written on the board there are expressions that define numbers. But I am mistaken about my whereabouts -- I am in fact in room 101. It is clear that A and B define numbers. But what number does C define? We can reason as follows:

> Suppose towards a contradiction that C defines a number, say k. Then the sum of the numbers defined by A, B and C is $\pi+6+k$. But this number is the number defined by C; so $k=k+\pi+6$, which is a contradiction. So C is a pathological defining expression: it appears to define a number, but it does not, on pain of contradiction.

From this reasoning, we infer:

(1) C is pathological, and does not define a number.

We can now strengthen our reasoning, building on our conclusion that C is pathological. We can argue as follows:

> Suppose that C is indeed pathological, and does not define a number. Then A and B are the only expressions on the board that define numbers. So, the sum of the numbers defined by expressions on the board in room 101 at noon 7/1/95 is π+6. But in the previous sentence there occurs a token of the same type as C, call it C′. But C′ defines π+6. Now we infer:

(2) C′ - a token of the same type as C - defines π+6.

This reasoning certainly calls for explanation. Since C and C′ are composed of the very same words with the very same linguistic meaning, it is natural to suppose that they define the same number. But then C, like C′, will define π+6. And this means that C at first has no denotation, and then acquires one. Compare the Richard case: just as the sentence G had no denotation and then acquired one, so here the phrase C at first has no denotation, and then acquires one. In both cases, once we declare the problematic phrase pathological, we are left with the unproblematic phrases that do define numbers - and this at once provides a denotation for the problematic phrase.

If we follow Poincaré's line, we will say that in the present case the classification is not immutable: the number π+6 is first in the category of numbers not defined by expressions on the board, and is subsequently transferred to the category of numbers that <u>are</u> defined by expressions on the board. So the classification is non-predicative, and we are to reject C on these grounds. But as we have seen, there is a problem with this line of Poincaré's. The old difficulty remains: our very rejection of C does not avoid the natural inference to (2), and the return of the paradox.

But there is an alternative. The reasoning to (2) is natural and intuitive, and appears to be valid. We should not block the reasoning by artificial, <u>ad hoc</u> means. Rather, we should find a plausible analysis that preserves the validity of the reasoning. Let us treat the modification of the classification not as a sign of pathology but rather as the result of valid reasoning.

Since there appear to be no *semantic* differences between the tokens C and C′, it is natural to look for *pragmatic* differences. There are a number of differences between the context of C and the context of C′. Clearly there are differences of time and place; C′ is uttered later, and, unlike C, C′ is not written on the board in room 101. And if we suppose that someone other than myself carries out the reasoning, there is a difference of speaker too. Still, the familiar contextual parameters of speaker, time, and place do not tell the whole story.

A fourth difference is that C and C′ are embedded in different stages of the discourse. We may split the discourse into two stages, the first culminating in (1), and the second leading from (1) to (2). The token C belongs to the first stage, and the token C′ to the second. In general, the correct interpretation of an expression or a stretch of discourse may depend on the larger discourse in which it is embedded.[9] The reasoning from (1) to (2) is second not merely in temporal order, but in logical order as well. The second stage starts out from a subconclusion, namely (1), established by the first stage of the argument. We can think of the second stage as *reflective* with respect to the first: at the second stage of the reasoning, we reflect on the pathological nature of C, established by the first stage. This logical order constitutes a difference in the relation that each stage of the discourse bears to the discourse as a whole.

A fifth difference is found in speaker's intentions. When I first utter C, I do not intend to produce a pathological utterance, but rather to pick out the expressions written on the board next door. This intention is overridden, given the time and place of my utterance: I have unwittingly landed myself in paradox. Throughout the second stage of the reasoning, the reflective stage, we have a very different intention, to treat C as a pathological defining expression and see where this leads us. This intention is not overridden. On the contrary, the fact that we intentionally take C to be pathological leads to the conclusion that C′ defines a number.

There is a sixth difference between the two stages, a shift of relevant information. When I first utter C, the information that C is pathological is not available to me. But this information is available throughout the reflective second stage of the reasoning: it is precisely what is established by the reasoning at the first stage. The reasoning of the second stage should be interpreted as incorporating this information.[10]

So we distinguish two contexts, one in which I write C, and the second in which we produce C′. Call these *the original context* and *the reflective context* respectively. Between these contexts there is a shift in a number of contextual parameters: speaker, time, place, discourse position, intention, and relevant information. Now we want to explain the fact that C does not define a number (conclusion 1) while C′ does (conclusion 2), yet C and C′ appear to be semantically indistinguishable. A pragmatic explanation is indicated, one that takes account of the shift in the contextual parameters. If we accept the appropriateness of a pragmatic explanation, then we should expect to find a term occurring in C and C′, and in (1) and (2), that is context-sensitive. When we inspect the terms occurring in these expressions, there is only one plausible candidate: the predicate `defines'. I propose that, in the absence of any reasonable alternative, we take the predicate `defines' to be the context-sensitive term.

Let `$\text{defines}_{OR}$' abbreviate `defines in the original context', and let `$\text{defines}_{RE}$' abbreviate `defines in the reflective context'. Let us turn first to the representation of C. We take the occurrence of `defines' in C to be sensitive to the context in which it occurs.[11] So C is analyzed as:

(C) the sum of the numbers defined_{OR} by expressions on the board in room 101 at noon 7/1/95.

At the first stage, we go on to reason that C is pathological. In the course of this reasoning, there is an implicit use of what we can call a *definition schema*. A definition schema has this form:

s defines n iff p is identical to n,

where instances of the schema are obtained by substituting for `p' any defining expression, for `s' any name of this expression, and for `n' any name of an individual. We suppose that C defines k. The instantiation of the schema for C and k is:

C defines k iff the sum of the numbers defined by expressions on the board in room 101 at noon 7/1/95 is k.

In our reasoning, we assume the left hand side of the biconditional, infer the right hand side, and obtain a contradiction - since the sum of the numbers defined by A, B and C is $\pi+6+k$. But this contradiction is forthcoming only if the definition schema is the schema for the occurrence of `defines' in C. That is, the schema we have utilized in our reasoning is the `defines_{OR}' schema: we derive a contradiction via the biconditional

C defines_{OR} k iff the sum of the numbers defined_{OR} by expressions on the board in room 101 at noon 7/1/95 is k.

A definition schema provides definition conditions for a given referring expression - that is, the conditions under which the phrase defines an individual.[12] At the first stage of the reasoning, we give the definition conditions for defining expressions - in particular C - via the `defines_{OR}' schema. We find that C does not have definition_{OR} conditions, and conclude that C does not define_{OR} a number.

Now, at the second stage, we reason that A and B are the only expressions on the board that define_{OR} numbers, since C does not. A and B define π and 6 respectively. So we infer that the sum of the numbers defined_{OR} by expressions on the board in room 101 at noon 7/1/95 is π+6. In producing C′ here, we have in effect repeated C. But we have repeated C in a new reflective context, in which we no longer provide definition conditions via the `$\text{defines}_{OR}$' schema. Instead, we provide definition conditions for C′ via the `defines_{RE}' schema - and C′ <u>does</u> have definition_{RE} conditions. Both sides of the biconditional

> C′ defines_{RE} k iff the sum of the numbers defined_{OR} by expressions on the board in room 101 at noon 7/1/95 is k

are true for k = π+6. C′ defines_{RE} π+6.

C and C′ are indeed *semantically* indistinguishable. The difference between them is purely pragmatic. It is a matter of the definition schema by which C and C′ are given definition conditions. At the first stage of the reasoning, C is assessed via the `$\text{defines}_{OR}$' schema; at the second stage, C′ is assessed via the `defines_{RE}' schema. The schema that provides definition conditions is determined by the contextual parameters. And the shift in discourse position, intentions, and information produces a change of implicated schema. At the second stage, the schema that provides definition conditions is reflective with respect to the defining phrase C. That the schema is reflective in this way is a product of the reasoning of the first stage, the assessment of C as a pathological defining phrase, and the intention to treat C as such.

According to our analysis, then, (1) is represented by

> C does not define_{OR} a number,

and (2) by

> C′ defines_{RE} π+6.

(1) and (2) are both true, and are the result of valid reasoning. Notice that if we assess C via the `$\text{defines}_{RE}$' schema, we find that C, like C′, $\text{defines}_{RE}$ π+6; and if we assess C′ via the `defines_{OR}'

schema, we find that C′, like C, does not define$_{OR}$ a number. Both C and C′ have definition$_{RE}$ conditions; neither have denotation$_{OR}$ conditions. So the predicates `defines$_{OR}$' and `defines$_{RE}$' have different extensions. For C and C′ are not in the extension of `defines$_{OR}$' (more precisely, neither C nor C′ are the first member of any ordered pair in the extension of `defines$_{OR}$'). But C and C′ are in the extension of `defines$_{RE}$' (more precisely, the ordered pairs <C,π+6> and <C′,π+6> are in the extension of `defines$_{RE}$'). So `defines' is a context-sensitive term that shifts its extension according to context.[13]

In this way, we account for the phenomenon that Poincaré makes explicit: that C first has no denotation, and then acquires one. According to our analysis, C does not define$_{OR}$ a number - it does not define a number in the original context. But C does define$_{RE}$ a number - it does define a number when evaluated from the reflective context. For Poincaré, this shift is a symptom of pathology - the classification of numbers defined by expressions on the board is to be declared non-predicative. But as we saw in connection with Richard's paradox, this declaration does not avoid the modifications anyway. In contrast, if we follow the contextual line, we distinguish the numbers that are defined$_{OR}$ by expressions on the board, and the numbers that are defined$_{RE}$ by expressions on the board. We are not confronted with an unacceptably unstable classification - rather we have a context-sensitive definition predicate that shifts its extension according to context. On the contextual approach, we let the modifications happen. The contextual approach preserves the naturalness and the validity of the reasoning about C.

In exactly parallel fashion, we can present the reasoning associated with Richard's paradox in two stages. At the first stage we argue as follows:

> Obtain an enumeration of all the English phrases that define a real number, and let E be the set of numbers defined by these phrases. Now *let p be the digit in the nth decimal place of the nth number of the set E; let us form a number having 0 for its integral part and, in its nth decimal place, p+1 if p is not 8 or 9, and 1 otherwise*. Since the underlined sentence G is a diagonal definition, N is different from itself, and we have a contradiction. We infer:

(1^R) G is a pathological denoting phrase, and does not define a real.

At the second stage we argue:

Since G is a pathological denoting phrase, it is not among the phrases of English that

> define real numbers. Once the problematic phrase G is eliminated, we are left with the English phrases that do define real numbers. So now *let p be the digit in the nth decimal place of the nth number of the set E; let us form a number having 0 for its integral part and, in its nth decimal place, p+1 if p is not 8 or 9, and 1 otherwise*. The italicized sentence in the previous sentence, call it G′, is of the same type as R. We infer:

(2^R) G′ - a token of the same type as G - does define a real.

The analysis of this reasoning follows the same pattern as before.[14] Again, we distinguish two contexts, the original context in which G is produced, and the subsequent reflective context in which G′ is produced. We treat `defines' as a context-sensitive term. Let `defines_O' and `$\text{defines}_R$' abbreviate `defines in the original context' and `defines in the reflective context' respectively. Then the occurrence of `defines' in both G and G′ is represented by `defines_O'. (1^R) is represented by

G does not define_O a number,

and (2^R) is represented by

G′ defines_R a number.

In its context of utterance, G is assessed via the `$\text{defines}_O$' schema; and G does not have $\text{definition}_O$ conditions. In its context of utterance, G′ is assessed via the `defines_R' schema. G′ does have definition_R conditions, and does define_R a number (as does D). Again, we treat `defines' as a context-sensitive term, shifting its extension according to context.

I am, then, proposing a contextual solution to the definability paradoxes. Contextual approaches to the Liar paradox and the concept of truth have received increasing support in recent years.[15] What I am suggesting here is the same kind of solution to the definability paradoxes. These paradoxes give rise to phrases that at first have no denotation and then acquire one, and to numbers that are first classified in one way and then in another. For Poincaré, such shifting, mutable classifications must be avoided. On the present proposal, these shifts occur in the course of valid reasoning, and signal the context-sensitivity of the definition predicate.

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Endnotes

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1. We should note in passing two easily corrected flaws in Richard's presentation. Though he refers to E as a set, he treats it as a sequence. And he overlooks the fact that different permutations of letters may define the same number. Here is one way to correct the second flaw: for any number that is defined by more than one permutation, we choose the permutation that is first in alphabetical order. Such a procedure is found in Poincaré: "In order to classify the integers, or the points in space, I shall consider the sentence which defines each integer or each point. Since it can happen that the same number or the same point can be defined by many sentences, I shall arrange these sentences in alphabetical order and I shall choose the first among these" (1909, p. 48).

2. See also p. 316. According to Poincaré, a proof that is truly based on logical principles is composed of a series of propositions, where those serving as premises are identities or definitions, and the rest are deduced step by step from the premises. Proofs do not yield new truths - they appear to do so only because each step of the proof may not be made fully explicit. If we replace the various expressions that appear in a proof by their definitions, and trace back through the steps, all that remains will be identities, which reduce to "a huge tautology" (p. 316). Poincaré concludes: "La Logique reste donc sterile ..." (*ibid*.)

Why, asks Poincaré, might we think otherwise? Suppose that some of the definitions are non-predicative, and contain a vicious circle. Then there will be no reduction to a tautology. If we fail to recognize the problematic nature of these definitions, we may be misled into thinking that a proof does yield new truths, that logic is *not* sterile. But if we admit non-predicative definitions, says Poincaré, then what we get are not new truths, but paradoxes: "Dans ces conditions, la Logistique n'est plus sterile, elle engendre l'antinomie" (*ibid*.).

3. Poincaré regarded his way out as an endorsement of Richard's proposal (see Poincaré 1906, p. 305 and p. 307). In his 1905 letter, Richard not only presented his paradox, but also offered a way out. In a later paper, Richard reconstructed his cryptic proposal in a little more detail:

> It seemed to me easy enough to explain this paradox. Let G be the phrase that defines N. This phrase is an arrangement of words. Since the elements of E come from arrangements of words, in forming the set E we will encounter the phrase G. Suppose we encounter it at rank p. At this moment it does not have meaning, for at this moment the first p-1 elements of E are the only ones defined. Having no meaning, the phrase G must be crossed out. (p. 95)

According to Richard, only finitely many members of E are defined by the time we reach G, so that G contains a term, viz. `E', that is not totally defined. And so we must cross G out. Poincaré provides his own twist to Richard's proposal: we must cross G out to avoid a vicious circle in the definition of E. (For more on Richard's proposal and Peano's objections to it, see Simmons 1994a)

4. Notice that we are assuming here that G is the only problematic definition in our list. Since

there will be many other problematic definitions, this is clearly an over-simplification. However, it is implicitly assumed by Poincaré (and Richard) in their treatments of the paradox, and we shall accept it for the sake of our discussion.

5. This difficulty is pointed out by Peano. Peano writes:

> If the phrase that defines N does not express a number, as was demonstrated above, then, when I calculate N, I pass by this phrase, which does not define a number, and the definition of N acquires a meaning. That is to say, if N does not exist, then it exists. (Peano 1906, p. 357).

For more on Peano's objection, and his own treatment of the paradox, see Simmons 1994a.

6. Richard is also sensitive to this process: he describes its initial stages in the final paragraph of his 1905 letter. See Simmons 1994b for more on this.

7. For Poincaré, a definition is a kind of classification. Poincaré writes: "Every definition is, in effect, a classification. It separates the objects which satisfy the definition from those which do not, and it arranges them in two distinct classes. ... A definition, like all classifications, may or may not be predicative" (Poincaré 1909, p. 48).

8. Traditionally, we speak of the paradoxes of *definability*. But I think the traditional terminology is misleading in two ways. First, the appearance of the modal element is deceptive - in the presentation of Richard's paradox, every occurrence of `definable' may be replaced by `defined'. Second, the paradoxes do not turn on any technical sense of definition. When we generate these paradoxes, we count as a definition any phrase that denotes or refers to a number; so, for example, the phrase `the number of planets' will count as a definition of the number 9. The paradoxes turn on the semantic relation that holds between a referring expression and its referent, whether the relation is expressed by `defines' or `denotes' or `refers to'. The paradoxes would be better called paradoxes of reference, or paradoxes of denotation. Though I shall keep to the traditional terminology, my use of the term 'defines' could be replaced throughout by the term 'refers to' or the term 'denotes'.

9. This has been emphasized by Paul Ziff, among others; see Ziff 1972, p. 36.

10. A shift of relevant information may not always occur in strengthened reasoning: I may intentionally produce a pathological denoting expression, in order to reflect upon it.

11. Here and elsewhere I speak of an occurrence of `defines' in an utterance, even if that occurrence is in the passive form.

12. Compare the more familiar truth schema:

s is true iff p,

where to obtain an instance we substitute for `p' a sentence, and for `s' a name of that sentence. The used sentence on the right hand side provides truth conditions for the mentioned sentence on the left.

13. See Simmons 1994b for more on this contextual approach to the definability paradoxes. According to the 'singularity' account I develop, the predicates '$defines_{OR}$' and '$defines_{RE}$' apply almost everywhere, except for certain singular points (for example, C is a singularity of '$defines_{OR}$'). Neither predicate is more comprehensive than the other - my account is strongly anti-hierarchical, rejecting any Tarskian stratification of the definition predicate. I offer a similar account of truth and the Liar paradox in Simmons 1993. Both accounts are in the spirit of brief remarks in Gödel 1944.

14. We have presented the reasoning in terms of tokens. Alternatively one could present the reasoning in terms of the sentence type of G, associated first with the original context, and subsequently with the reflective context. The analysis of the reasoning would be essentially unchanged.

15. See Simmons 1993, and also Parsons 1974, Burge 1979, Barwise and Etchemendy 1987, and Gaifman 1988 and 1992.